Transmission and reflection coefficients for a scalar field inside a charged black hole

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Using the late-time expansion we calculate the leading-order coefficients describing the evolution of a massless scalar field inside a Reissner-Nordstrom black hole. These coefficients may be interpreted as the reflection and transmission coefficients for scalar-field modes propagating from the event horizon to the Cauchy horizon. Our results agree with those obtained previously by Gursel et al by a different method.

In a recent paper [1], we analysed the evolution of a massless scalar field inside a Reissner-Nordstrom black hole. The analysis was based on a new method, the *late-time expansion*, which is essentially an expansion in the small parameter $1/t$. This method takes advantage of the relatively simple form of the ingoing radiation tails, which are known to obey an inverse-power law. [2] Assuming initial data $\psi \cong v^{-n}$ at the event horizon (EH), the asymptotic behavior near the Cauchy horizon (CH) was found in Ref. [1] to be

$$\psi = \sum_{j=0}^{\infty} \left[a_j \, u^{-n-j} \; + \; b_j \, v^{-n-j} \right] + O\left(\delta r\right) , \tag{1}$$

where $\delta r \equiv r - r_-$, u and v are the double-null Eddington-Finkelstein-like coordinates, and a_j and b_j are constants (the notation here is the same as that of Ref. [1], except for a few changes specified below). Of special importance are the coefficients a_0 and b_0, which dominate the evolution near the CH at late time (i.e. $v \rightarrow \infty$, $u \rightarrow -\infty$). These parameters may be interpreted (in a somewhat vague sense) as the reflection and transmission coefficients, respectively, for low-frequency scalar-field modes propagating from the EH to the CH. The explicit value of these coefficients (like all other coefficients a_j and b_j) was not calculated in Ref. [1]. In this paper we shall use the late-time expansion to calculate a_0 and b_0. The problem of a massless scalar field inside a charged black hole was previously analysed [3–5] by a different method (Fourier integration in the complex plain), and the transmission and reflection coefficients were calculated in Ref. [4]. Our goal here is to compare the results of the late- time expansion to those obtained in Ref. [4] by the other method.

We shall use here the same notation as in Ref. [1], except for the following changes:
(i) We denote here the "tortoise" coordinate by r^* (not x);
(ii) κ_-, the inner horizon's surface gravity, is defined here in the standard way, $\kappa_- = (r_+ - r_-)/(2r_-^2)$;
(iii) We denote the total scalar field by Ψ (we reserve the symbol ψ for the spherical-harmonics modes of the field). Ψ satisfies the Klein-Gordon (KG) equation, $\Psi_{;\alpha}^{;\alpha} = 0$.

We first decompose Ψ into spherical harmonics:

$$\Psi(r,\theta,\varphi,t)=\sum_{lm} Y_{lm}(\theta,\varphi)\,\psi_{lm}(r,t)\ .$$

For brevity, we shall omit the indices l,m, and denote ψ_{lm} by ψ. We then use the late-time expansion:

$$\psi=\sum_{k=0}^{\infty}\psi_k(r)t^{-n-k}\ . \tag{2}$$

Here, like in Ref. [1], when studying the asymptotic behavior at the CH, we restrict attention to the leading-order of ψ (and ψ_k) in $\delta r\equiv r-r_-$. At this order, the functions ψ_k are polynomials in r^* [1], of order k: [6]

$$\psi_k\cong\sum_{i=0}^{k} c^{ki}\,r^{*i}\ . \tag{3}$$

From the analysis in Ref. [1] it is obvious that the leading- order term in Eq. (1), $a_0u^{-n}+b_0v^{-n}$, is equal to $t^{-n}F^1(w)$ (i.e. the term $j=1$ in Eq. (29) there [7]), namely,

$$a_0u^{-n}+b_0v^{-n}=t^{-n}\left[c^{0,0}+c^{1,1}(r^*/t)+c^{2,2}(r^*/t)^2+...\right]\ . \tag{4}$$

Therefore, the coefficients a_0 and b_0 depend only on the parameters $c^{k,k}$. We shall now show that a_0 and b_0 can be determined directly from $c^{0,0}$ and $c^{1,1}$. For simplicity, let us define

$$\hat{u}\equiv -u\ \ ,\ \ \hat{a}_0\equiv(-1)^n a_0\ ,$$

such that $a_0u^{-n}=\hat{a}_0\hat{u}^{-n}$. Recalling that $\hat{u}=t-r^*\ \ ,\ \ v=t+r^*$, we may write

$$a_0u^{-n}+b_0v^{-n}=t^{-n}\left[\hat{a}_0(1-r^*/t)^{-n}+b_0(1+r^*/t)^{-n}\right]$$
$$=t^{-n}\left[(\hat{a}_0+b_0)+n(\hat{a}_0-b_0)\,(r^*/t)+\frac{n(n+1)(\hat{a}_0+b_0)}{2}\,(r^*/t)^2+...\right]\ . \tag{5}$$

Comparing Eqs. (4) and (5), we find

$$c^{0,0}=\hat{a}_0+b_0\quad ,\quad c^{1,1}=n(\hat{a}_0-b_0)\ . \tag{6}$$

Therefore, all we need to do is to calculate the coefficients $c^{0,0}$ and $c^{1,1}$. This, in turn, requires the solution of the differential equation for $\psi_{k=0}$ and $\psi_{k=1}$. Recall that for that purpose it

is not sufficient to consider the leading order of these functions in δr: Since the parameters $c^{0,0}$ and $c^{1,1}$ will be determined by matching the functions ψ_0 and ψ_1 to the initial data at the EH, we shall need the exact form of these functions in the entire range $r_- < r < r_+$.

The two functions ψ_0 and ψ_1 satisfy the same "static" (i.e. t-independent) KG equation,

$$f\psi_{k,rr} + (f_{,r} + 2f/r)\,\psi_{k,r} - \frac{l(l+1)}{r^2}\psi_k = \ 0 \qquad (k = 0, 1)\ ,$$

where $f \equiv 1 - 2M/r + e^2/r^2$. The general solution of this equation is [8]

$$\psi_k = a\,Q_l\ (x) + b\,P_l\ (x)\ \ ,$$

where a and b are arbitrary constants, P_l and Q_l are the Legendre functions of the first and second kinds, respectively, and

$$x \equiv \frac{2r - r_+ - r_-}{r_+ - r_-}\ .$$

Thus, the most general expressions for ψ_0 and ψ_1 are

$$\psi_0 = a^0\,Q_l\ (x) + b^0\,P_l\ (x)$$

and

$$\psi_1 = a^1\,Q_l\ (x) + b^1\,P_l\ (x)\ \ . \tag{7}$$

The parameters a^0, b^0, a^1 and b^1 are to be determined from the initial conditions at the EH. The event and inner horizons correspond to $x = 1$ and $x = -1$, respectively. At both points, $P_l(x)$ is regular and $Q_l(x)$ diverges logarithmically. The asymptotic form of the two Legendre functions at the horizons is given by

$$P_l\,(x = 1) = 1 \quad , \quad Q_l(x) = -\frac{1}{2}\ln(1-x) + regular\ term \qquad (x \to 1) \tag{8}$$

and

$$P_l\,(x = -1) = (-1)^l \quad , \quad Q_l(x) = \frac{(-1)^l}{2}\ln(1+x) + \ regular\ term \qquad (x \to -1)\ . \tag{9}$$

As was shown in Ref. [1], regularity at the EH implies $a^0 = 0$, so

$$\psi_0 = b^0 \, P_l \, (x) \ . \tag{10}$$

Comparing Eqs. (3), (9) and (10), we find

$$c^{0,0} = (-1)^l \, b^0 \ . \tag{11}$$

Also, since both r^* and $Q_l(x)$ diverge logarithmically at the CH, $c^{1,1}$ must be proportional to a^1. Since at the CH $r \, - r_- \, \propto e^{-2\kappa_- r^*}$ and

$$1 + x = \frac{2}{r_+ - r_-} \, (r - r_-) \, ,$$

we have

$$\ln(1+x) = -2\kappa_- r^* + \ \textit{regular term} \qquad (x \to -1) \ . \tag{12}$$

We find from Eqs. (3,7,9,12)

$$c^{1,1} = (-1)^{l+1} \, \kappa_- \, a^1 \ . \tag{13}$$

We shall now calculate b^0 and a^1 from the initial data at the EH. Presumably, we have there $\psi = v^{-n}$, namely,

$$\psi = t^{-n} \, [1 + r^*/t]^{-n} = t^{-n} \, [1 - n(r^*/t) + ...(\, r^*/t)^2 + ...] \ . \tag{14}$$

It is obvious from Eq. (2) that the term proportional to $(r^*/t)^k$ in the brackets at the right-hand side will come from ψ_k. Considering first the contribution from $\psi_{k=0}$, we find from Eqs. (8,10,14)

$$b^0 = 1 \ . \tag{15}$$

Consider next the contribution from $k = 1$. At the EH, too, $Q_l(x)$ is proportional to r^* as both diverge logarithmically. In analogy with the above treatment of the CH, we now have

$$\ln(1-x) = 2\kappa_+ r^* + \ \textit{regular term} \qquad (x \to 1) \ .$$

Equations (7) and (8) then yield

$$\psi_1 = -\kappa_+ a^1 \, r^* + \; regular \; term \qquad (x \to 1) \; ,$$

so Eq. (14) implies

$$a^1 = n/\kappa_+ \; . \tag{16}$$

Returning to the asymptotic behavior at the CH, we find [cf. Eqs. (11,13,15,16)]

$$c^{0,0} = (-1)^l \quad , \quad c^{1,1} = (-1)^{l+1} \, n\kappa_- / \kappa_+ \; ,$$

and therefore

$$\hat{a}_0 = (-1)^l \; \frac{\kappa_+ - \kappa_-}{2\kappa_+} \quad , \quad b_0 = (-1)^l \; \frac{\kappa_+ + \kappa_-}{2\kappa_+} \tag{17}$$

[cf. Eq. (6)]. Substituting $\kappa_\pm = \frac{r_+ - r_-}{2r_\pm^2}$, and recalling that $\hat{a}_0 \equiv (-1)^n a_0$, we finally obtain the desired expression for the reflection and transmission coefficients:

$$a_0 = (-1)^{l+n} \; \frac{r_-^2 - r_+^2}{2r_-^2} \quad , \quad b_0 = (-1)^l \; \frac{r_-^2 + r_+^2}{2r_-^2} \; . \tag{18}$$

This result is the same as the one obtained by Gursel et al [4] by a different method (Fourier integration in the complex plain). [9]

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[6] Equation (24) of Ref. [1] describes a polynomial of order $k+1$. However, as was shown in section VII there, regularity at the EH implies $c^{0,1}=0$. Since $c^{1,2}$ vanishes too, it follows from the recursion relation for ψ_k [Eqs. (13,22) there] that $c^{k,k+1}$ vanishes for all k.

[7] Note that regularity at the EH implies $F^0(w)=0$, because $c^{k,k+1}=0$ [6], so the leading-order term in Eq. (29) of Ref. [1] is j=1.

[8] This is essentially the "$k=0$ solution" in Ref. [4] [cf. Eq. (6) there], except that ψ there is r times our ψ.

[9] See Eqs. (10,11,15) in [4], and note the following differences in terminology: (i) The coefficient related to transmission is denoted b_0 here and $A(0)$ in Ref. [4], whereas the one related to reflection is denoted a_0 here and $B(0)$ in Ref. [4]; (ii) The field denoted there by ψ is the KG field multiplied by r (whereas here Ψ is the KG field itself); (iii) In Ref. [4] it is presumed that $n\equiv 2l+2$ (which is the case if the mode has a nonvanishing static initial multipole moment [2]), so n is always even. In our analysis n may be any integer. (Recall that for a mode with a vanishing static initial multipole moment, one expects $n=2l+3$ [2].)